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Quasi-projective surfaces with finite π_1 at infinity

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Introduction

C. P. Ramanujam's theorem on characterization of $\mathbb{C}^2$ can be stated as follows.

'Let V be an affine, non-singular, rational surface/$\mathbb{C}$ such that (i) $\Gamma(V)$ is a U.F.D. (ii) $\Gamma(V)^* = \mathbb{C}^*$ and (iii) the fundamental group at infinity of V is trivial. Then $V \approx \mathbb{C}^2$ as an affine variety.'

In [2], this result was generalized by assuming that the fundamental group at infinity of V is finite. Then together with (i) and (ii) above, V is still isomorphic to $\mathbb{C}^2$. For singular affine surfaces also, the following result holds, see [3].

'Let V be a normal, affine surface which is topologically contractible and has finite fundamental group at infinity. Then $V \approx \mathbb{C}^2/G$, where G is a finite subgroup of $GL(2,\mathbb{C})$.'

On the other hand, M. Miyanishi, T. Sugie and T. Fujita proved the following.

'Let V be an affine, non-singular surface satisfying (i) $\Gamma(V)$ is a U.F.D. (ii) $\Gamma(V)^* = \mathbb{C}^*$ and (iii) $\bar{\kappa}(V) = -\infty$.

Then $V \approx \mathbb{C}^2$ as an affine variety.'

The theory of logarithmic Kodaira dimension has proved to be very important for studying non-complete surfaces. Our aim in this paper is give a relationship between the topological method of C. P. Ramanujam and the geometric method of Miyanishi, Sugie, Fujita. Our result is the following.

Theorem: Let V be a non-singular, affine surface/$\mathbb{C}$ which has finite fundamental group at infinity. Then $\bar{\kappa}(V) = -\infty$.

See §1 for a slight generalization of this result.

We will give two different proofs of this results, one á la Ramanujam method and the other using T. Fujita's results in [1]. In both proofs, a result of A. R. Shastri on the classification of trees of $\mathbb{P}^1$'s having finite local fundamental group plays a crucial role. Shastri's proof depends on C. P. Ramanujam's method plus a concept from 3-dimenisonal topology. We hope that a more geometric method can be found to eliminate the use of 3-dimensional topology.

Shastri proved in [6] that an affine, normal surface with finite fundamental group at infinity in rational. Thus our result implies more. There exist easy examples of affine surfaces with $\bar{\kappa} = -\infty$ but non-finite π_1 at infinity.

§1. Shastri's Theorem.

We begin with some notations. For any positive integers, $0 < \lambda < n$ such that $(n,\lambda) = 1$, let $\langle n,\lambda\rangle$ denote the negative definite linear tree $\overset{-a_1}{*}\!\!-\!\!-\!\!\overset{-a_2}{*}\!\!-\!\!-\!\!*\cdots\cdots\overset{-a_k}{*}$ where $a_i \geq 2$ are integers defined by $n/\lambda = a_1 - \cfrac{1}{a_2 - \cfrac{\ddots}{\ddots - \cfrac{1}{a_k}}}$.

Let $\langle\langle n,\lambda\rangle\rangle$ denote the tree $\overset{0}{*}\!\!-\!\!-\!\!\overset{0}{*}\!\!-\!\!-\!\!\overset{-a_1}{*}\cdots\cdots\overset{-a_k}{*}$.

For n_j , λ_j as above define a_{ji} by using the continued fraction expansion of n_j/λ_j . For any $a \in Z$, let $\langle a\,;\, n_1 , \lambda_1 \,;\, n_2 , \lambda_2 \,;\, n_3 , \lambda_3\rangle$ denote the tree

$$\begin{array}{ccccccccc}
\overset{-a_{1k_1}}{\circ} & \cdots\cdots & \overset{-a_{11}}{\circ} & \text{---} & \overset{-a}{*} & \text{---} & \overset{-a_{21}}{*} & \cdots\cdots & \overset{-a_{2k_2}}{*} \\
 & & & & \vdots & & & & \\
 & & & & * & & & & \\
 & & & & \vdots & & & & \\
 & & & & *\,-a_{3k_3} & & & &
\end{array}$$

The result of Shastri mentioned in the introduction is the following.

Theorem 1: Let B be a normal, quasi-projective surface with a compactification $V \subset \bar{V}$ such that $\bar{V}$ is non-singular along $\bar{V} - V$. Suppose the divisor $D = \bar{V} - V$ has simple normal crossings, D is connected and the fundamental group at infinity of V is finite. Then the dual graph of D (each irreducible component of D is isomorphic to $\mathbb{P}^1$) is equivalent to one of the following trees (equivalence via blowing-ups and downs)

(i) The empty tree or $\overset{0}{*}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{*}$

(ii) $<n,\lambda>$

(iii) $<a\, ;\, 2\, ,\, 1\, ;\, n_2\, ,\, \lambda_2\, ;\, n_3\, ,\, \lambda_3>$ where $\{n_2\, ,\, n_3\}$ is one of the pairs $\{3,3\}$, $\{3,4\}$, $\{3,5\}$ or $\{2,n\}$ for any $n \geqq 2$, $0 < \lambda_i < n_i$ with $(n_i\, ,\, \lambda_i) = 1$ and $a \geqq 2$.

(iv) The trees mentioned in (iii) except that $a \leqq 1$.

(v) The trees $T^{(\nu)}$ where T is one of the trees in (ii) or (iii) and $\nu \in T$ is any vertex.

Here $T^{(\nu)}$ denotes the tree obtained from T by adding two more vertices $\nu_1\, ,\, \nu_2$ with weights at $\nu_1\, ,\, \nu_2$ both 0 and two more links $[\nu\, ;\, \nu_1]$ and $[\nu_1\, ;\, \nu_2]$ for any vertex ν of T .

Affine surfaces of the form $\mathbb{C}^2/G$, where G is a finite subgroup of $GL(2,\mathbb{C})$ have finite fundamental group at infinity. In this case, the configuration of curves at infinity is given by the following result of Shastri.

Theorem 2: For $V \approx \mathbb{C}^2/G$, the dual graph of $\bar{V} - V$ is equivalent to one of the following

(i) $\overset{0}{*}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{*}$ if $G = (e)$.

(ii) $\ll n,\lambda \gg$ if $G \approx Z/(n)$, λ depending on the inclusion $Z/(n) \hookrightarrow GL(2,\mathbb{C})$.

(iii) In all other cases, the tree is $<a\,;\,2\,,\,1\,;\,n_2\,,\,\lambda_2\,;\,n_3\,,\,\lambda_3>$ with $a \leqq 1$ and $n_i\,,\,\lambda_i$ as in Theorem 1 (iii).

We will use these two results of Shastri to give a proof of the main result of this paper

Theorem. Let V be a non-singular, quasi-projective surface which is connected at infinity and has finite fundamental group at infinity. Suppose $\bar{V} - V$ supports an effective divisor Δ with $\Delta^2 > 0$. Then $\bar{K}(V) = -\infty$ (Here $\bar{V}$ is a projective compactification of V which is smooth along $\bar{V} - V$).

Proof. We can assume that the dual graph of $\bar{V} - V$ has only simple normal crossings. Let $\bar{V} - V = \bigcup_{i=1}^{r} C_i$ where C_i are irreducible components. For a suitable tubular neighbourhood T of $\bigcup_{i=1}^{r} C_i$, the boundary ∂T is a C^∞ compact 3-manifold, $C = \bigcup_{i=1}^{r} C_i$ is a strong deformation retract of T and T is obtained by the process of "plumbing". For a precise definition, see [4].

By assumption $\pi_1(\partial T)$ is finite. This implies easily that each $C_i \approx \mathbb{P}^1$ and the dual graph of $\bigcup_{i=1}^{r} C_i$ is a tree. Also ∂T is a strong deformation retract of $T - C$. Thus $\pi_1(T - C)$ is finite. Let $\tilde{T}'$ be the universal covering of $T - C$ with $\tilde{T}' \xrightarrow{\phi} T - C$ the covering map. Then $\tilde{T}'$ is a complex manifold and ϕ is a holomorphic, proper map (with finite fibres). By Grauert-Remmert's theorem, we can embed $\tilde{T}' \subset \tilde{T}$ where $\tilde{T}$ is a normal complex space such that $\tilde{T} - \tilde{T}'$

is a finite union of compact analytic curves. Further we can assume that $\tilde{T}$ is smooth, ϕ extends to a proper holomorphic map $\tilde{T} \longrightarrow T$, which we still call ϕ. By resolving singularities, we can assume that the curve $\tilde{T} - \tilde{T}'$ has simple normal crossings.

By construction, $\pi_1(\partial\tilde{T})$ is trivial. This implies that each irreducible component of $\tilde{T} - \tilde{T}'$ is isomorphic to $\mathbb{P}^1$ and the dual graph of $\tilde{T} - \tilde{T}'$ is a tree. $\tilde{T}$ is also obtained by plumbing from $\tilde{C} = \tilde{T} - \tilde{T}'$.

Now we use Shastri's Theorem 1. Since C supports a divisor Δ with $\Delta^2 > 0$, the dual graph of C can be assumed to be $\overset{0}{*}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{*}$ or as in (iv) or (v) of Theorem 1. First assume that the graph is $\overset{0}{*}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{*}$ or $T^{(v)}$. Then $\exists$ curves C_1, C_2 in C s.t. $C_1^2 = 0 = C_2^2$, $C_1 \cdot C_2 = 1$ and C_1 meets no other curve in C except C_2. Then $(K+C)\cdot C_1 = -1$. This forces $|n(K+C)| = \phi$ for all $n \geqq 1$.

So we can assume that the dual graph of C is as in (iv) of Theorem 1.

Lemma 1. We can obtain $\tilde{C}$ from a single non-singular rational curve L with $L^2 = 1$, by a sequence of blowing ups and downs.

Proof. Let $G = \pi_1(\partial T) = \pi_1^{\infty}(V)$. We know that the dual graph of C is $\langle a\,;\,2\,,\,1\,;\,n_2\,,\,\lambda_2\,;\,n_3\,,\,\lambda_3\rangle$ as in (iv) of Theorem 1. From Theorem 2, we see that $\exists$ a normal, affine surface $W \approx \mathbb{C}^2/G$ where G has an embedding in $GL(2,\mathbb{C})$, and the dual graph of the infinity of W is same as that of V. But once the intersection matrix $(C_i \cdot C_j)$ of non-singular rational curves is given, the plumbing process gives the tubular neighbourhood T uniquely. Thus T is C^{∞}-diffeomorphic to a tubular nbd. N of the divisor at infinity D for W in a natural way. Then the universal covers of $T-C$ and $N-D$ are also diffeomorphic and the process of constructing $\tilde{T}$ from $\tilde{T}'$ being purely topological, we see that the dual graphs of $\tilde{C}$ and $\tilde{D}$ are naturally isomorphic with the corresponding components actually complex analytically isomorphic. But since the dual graph of $\tilde{D}$ can be obtained from a single non-singular rational curve M with $M^2 = 1$, the same is true about $\tilde{C}$. This proves Lemma 1.

Now let U be a complex manifold of dimension 2 which contains a $\mathbb{P}^1$ as a complex submanifold M with $M^2 = 1$. Then $|n(K_U + M)|$ has no sections for $n \geqq 1$. It follows easily that if $\tilde{U} \xrightarrow{\pi} U$ is a sequence of blowing-ups at points lying on M and $\pi^{-1}(M) = \tilde{M}$, then $|n(K_{\tilde{U}} + \tilde{M})|$ has no sections for $n \geqq 1$.

Suppose $|n(K_V + C)|$ has a non-zero section s. Since ϕ is a proper map, it follows that s gives a non-zero section of $|n(K_{\tilde{T}} + \tilde{C})|$. This follows easily from the Logarithmic Ramification Formula proved in [5].

This contradicts the observation above, completing the proof of our Theorem.

Remark. If we can find a direct argument for Lemma 1, then the use of 3-dimensional topology (which is used in Shastri's results) can be avoided.

§2. Another proof.

We will use the theory of Zariski decomposition of pseudo-effective divisors as discussed in Fujita's paper, [1]. The definitions of rational twig, bark of a tree etc. will be used as in [1].

Assume now that $\pi_1^\infty(V)$ is finite. As in the earlier proof, we have to only consider compactifications $V \subset \bar{V}$ s.t. the dual graph of $\bar{V} - V$ is $\langle a ; 2 , 1 ; n_1 , \lambda_1 ; n_2 , \lambda_2 \rangle$ as in Theroem 1, (iv). Assume that $\bar{\kappa}(V) \geqq 0$. Then $K + C$ is pseudo-effective.

Let $K + C = H + N$ be the Zariski-decomposition. We make two cases.

Case 1. Every irreducible component of $N = (K + C)^-$ is a component of C. Then the Lemma 6.17 in [1] implies that the dual graph Γ of $\bar{V} - V$ is an abnormal rational club. But the intersection matrix of an abnormal rational club is negative definite where as C supports an effective divisor with positive self intersection. So this case does not occur.

Case 2. Since Γ is not an abnormal rational club, $Bk(\Gamma) = Bk^*(\Gamma)$ by definition. If $N = Bk^*(\Gamma)$, then every irreducible component of N is a component of Γ and we get a contradiction as in case 1. Thus $N \neq Bk^*(\Gamma)$.

Now we can use Lemma (6.20) in [1]. There exists a component E of N which is an exceptional curve not in C satisfying one of the following conditions.

1) $C \cap E = \phi$.

2) $C \cdot E = 1$ and E meets a component of $Bk^*(C)$.

3) $C \cdot E > 1$ and E meets two components of C , one of which is a tip of a rational club of C .

3) cannot occur because C is connected. If 1) occurs, we can blow-down E without changing the fundamental group at infinity or $\bar{K}$. So we must consider case 2). We study the tree Γ more closely. $\Gamma = B + T_1 + T_2 + T_3$ where B is the unique curve which meets three other curves and $B^2 \geqq -1$. We can assume T_1 is the tree $\overset{-2}{*}$. Then T_2 has the form $\overset{-2}{*}$ or $\overset{-3}{*}$ or $\overset{-2}{*}\text{---}\overset{-2}{*}$ (we can assume that $d(T_2) = 2$ or 3). If T_2 is the tree $\overset{-2}{*}$, then T_3 can be any negative definite, minimal tree with determinant $n \geqq 2$. If $T_2 = \overset{-3}{*}$ or $\overset{-2}{*}\text{---}\overset{-2}{*}$ then T_3 is one of the negative definite minimal linear trees with determinant $3, 4$ or 5 .

Blow down E to get a surface $\bar{W}$, let $\bar{V} \xrightarrow{\pi} \bar{W}$ be the blowing down and $C' = \pi(C)$, $W = \bar{W} - C'$. Then C' looks like C (but may not be minimal). Then $W \subset V$, so it suffices to show that $\bar{K}(W) = -\infty$.

We can blow down exceptional curves of the 1st kind in C' to get a minimal tree which is either linear or has exactly one curve which is a branch point for the new tree. This way we get a new compactification of W , with the divisor at infinity $\tilde{C}$ having simple normal crossings. If one of the branches at the branch point has a non-negative weight, then the Corollary (6.14) in Fujita's paper implies that $\bar{K}(W) = -\infty$ and we are done. Similarly if $\tilde{C}$ is linear with a non-negative weight, $\bar{K}(W) = -\infty$.

We can thus assume that $\tilde{C}$ has exactly one branch point with three branches. From the nature of $\tilde{C}$, it is seen easily that the dual graph of $\tilde{C}$ is again of the type (iv) of Theorem 1. Also $\tilde{C}$ supports an effective divisor with positive self intersection. So we can again repeat the argument for W and in finitely many steps reach a Zariski-open subset of the original V with logarithmic Kodaira dimension $-\infty$.

The proof of the Theorem is complete.

References

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